



$$< \sum_{i=1}^n \frac{\varepsilon}{2} (x_i - x_{i-1}) + 2N\varepsilon \|\dot{\varphi}\| \quad (h(x) \leq 1)$$

$$< \frac{\varepsilon}{2} + 2N\varepsilon \frac{\varepsilon}{4N\varepsilon} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Since clearly  $S(h; \dot{\varphi}) \geq 0$  &  $\varepsilon > 0$  is arbitrary,

we have  $h \in \mathcal{R}[a, b]$  &  $\int_a^b h = 0$ . ~~✱~~

## § 7.2 Riemann Integrable Functions

### Thm 7.2.1 (Cauchy Criterion)

$f \in \mathcal{R}[a,b] \Leftrightarrow \forall \varepsilon > 0, \exists \eta_\varepsilon > 0$  such that

if  $\dot{\mathcal{P}}$  and  $\dot{\mathcal{Q}}$  are tagged partitions with

$$\|\dot{\mathcal{P}}\| < \eta_\varepsilon \text{ \& \ } \|\dot{\mathcal{Q}}\| < \eta_\varepsilon,$$

then  $|S(f, \dot{\mathcal{P}}) - S(f, \dot{\mathcal{Q}})| < \varepsilon$

(Compare :  $(x_n)$  converges  $\Leftrightarrow \forall \varepsilon > 0, \exists N_\varepsilon > 0$  s.t. if  $m, n \geq N_\varepsilon, |x_m - x_n| < \varepsilon$ )

Pf : ( $\Rightarrow$ ) If  $f \in \mathcal{R}[a,b]$  and  $L = \int_a^b f$ .

Then  $\forall \varepsilon > 0, \exists \eta_\varepsilon (= \delta_{\varepsilon/2}) > 0$  s.t.

if  $\|\dot{\mathcal{P}}\| < \eta_\varepsilon$  &  $\|\dot{\mathcal{Q}}\| < \eta_\varepsilon,$

then  $|S(f; \dot{\mathcal{P}}) - L| < \varepsilon/2$  and

$$|S(f; \dot{\mathcal{Q}}) - L| < \varepsilon/2.$$

$$\therefore |S(f; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{Q}})|$$

$$\leq |S(f; \dot{\mathcal{P}}) - L| + |L - S(f; \dot{\mathcal{Q}})|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ . (Done)}$$

( $\Leftarrow$ )

Step 1  $\exists$  seq.  $(\delta_n)$  with  $0 < \delta_{n+1} \leq \delta_n$ ,  $\forall n=1,2,3,\dots$

such that if  $\|\dot{\mathcal{P}}\| < \delta_n$  &  $\|\dot{\mathcal{Q}}\| < \delta_n$ ,

then  $|S(f, \dot{\mathcal{P}}) - S(f, \dot{\mathcal{Q}})| < \frac{1}{n}$ .

Pf of Step 1

By assumption, for  $\varepsilon = \frac{1}{n} > 0$ ,  $\exists \eta_{\frac{1}{n}} > 0$  s.t.

if  $\|\dot{\mathcal{P}}\| < \eta_{\frac{1}{n}}$  &  $\|\dot{\mathcal{Q}}\| < \eta_{\frac{1}{n}}$ ,

then  $|S(f, \dot{\mathcal{P}}) - S(f, \dot{\mathcal{Q}})| < \frac{1}{n}$ .

Let  $\delta_n = \min\{\eta_1, \eta_{\frac{1}{2}}, \dots, \eta_{\frac{1}{n}}\} > 0$ ,  $\forall n=1,2,3,\dots$

then  $\delta_{n+1} = \min\{\eta_1, \eta_{\frac{1}{2}}, \dots, \eta_{\frac{1}{n}}, \eta_{\frac{1}{n+1}}\}$

$$\leq \min\{\eta_1, \eta_{\frac{1}{2}}, \dots, \eta_{\frac{1}{n}}\} = \delta_n$$

And if  $\|\dot{\mathcal{P}}\| < \delta_n$  &  $\|\dot{\mathcal{Q}}\| < \delta_n$

then  $\|\dot{\mathcal{P}}\| < \eta_{\frac{1}{n}}$  &  $\|\dot{\mathcal{Q}}\| < \eta_{\frac{1}{n}}$

$\therefore |S(f, \dot{\mathcal{P}}) - S(f, \dot{\mathcal{Q}})| < \frac{1}{n}$  .  $\ast$

Step 2  $\exists$  a seq. of tagged partition  $\dot{\mathcal{Q}}_n$  s.t.

$\|\dot{\mathcal{Q}}_n\| < \delta_n$  and  $(\delta_n \text{ given in Step 1})$

$\lim_{n \rightarrow \infty} S(f, \dot{\mathcal{Q}}_n) = L$  exists.

Pf of Step 2:

For each  $n=1, 2, 3, \dots$ , choose any tagged partition

$$\dot{Q}_n \text{ with } \|\dot{Q}_n\| < \delta_n$$

(Existence of  $\dot{Q}_n$  is clear: for example, one may take

uniform partition with any tags  )

Then, for  $m \geq n \geq 1$ , we have

$$\|\dot{Q}_n\| < \delta_n \text{ and } \|\dot{Q}_m\| < \delta_m \leq \delta_n$$

$$\therefore \text{Step 1} \Rightarrow |S(f, \dot{Q}_n) - S(f, \dot{Q}_m)| < \frac{1}{n} \quad (*)$$

Hence  $\forall \varepsilon > 0$ , one can take any integer  $n_0 > \frac{1}{\varepsilon}$

and conclude that  $\forall m \geq n \geq n_0$ ,

$$(\Rightarrow \|\dot{Q}_m\| < \delta_m < \delta_{n_0} \text{ \& } \|\dot{Q}_n\| < \delta_n \leq \delta_{n_0})$$

$$|S(f, \dot{Q}_n) - S(f, \dot{Q}_m)| < \frac{1}{n_0} < \varepsilon$$

$\therefore (S(f, \dot{Q}_n))$  is a Cauchy sequence.

By completeness of  $\mathbb{R}$  (Thm 3.5.5 Cauchy Convergence Criterion),

$\lim_{n \rightarrow \infty} S(f, \dot{Q}_n)$  exists (let call it  $L$ )

~~✱~~

Final Step :  $f \in \mathcal{R}[a, b]$

Pf of Final Step

Using Step 2 and  $(*)_1$ , by taking  $n \rightarrow \infty$ , we have

$$|S(f, Q_n) - L| \leq \frac{1}{n}, \quad \forall n=1, 2, 3, \dots \quad \text{--- } (*)_2$$

Now  $\forall \varepsilon > 0$ , &  $k$  is an integer s.t.  $k > \frac{2}{\varepsilon}$ .

Then if  $\tilde{P}$  satisfies  $\|\tilde{P}\| < \delta_k$ ,

we have  $|S(f, \tilde{P}) - S(f, Q_k)| < \frac{1}{k}$  by Step 1

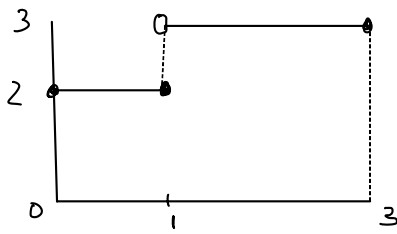
and hence

$$\begin{aligned} |S(f, \tilde{P}) - L| &\leq |S(f, \tilde{P}) - S(f, Q_k)| + |S(f, Q_k) - L| \\ &< \frac{1}{k} + \frac{1}{k} \quad (\text{by } (*)_2) \\ &= \frac{2}{k} < \varepsilon \end{aligned}$$

$$\therefore f \in \mathcal{R}[a, b] \quad \left( \text{ \& } \int_a^b f = L \right) \quad \#$$

Eg 7.2.2

(a)  $g: [0, 3] \rightarrow \mathbb{R}$  defined by  $g(x) = \begin{cases} 3, & 1 < x \leq 3 \\ 2, & 0 \leq x \leq 1 \end{cases}$



is (Riemann) integrable, Eg 7.1.4(b)

In Eg 7.1.4(b), we proved that

if  $\|\dot{P}\| < \delta$ , then

$$\delta - 5\delta \leq S(g, \dot{P}) \leq \delta + 5\delta.$$

If  $\dot{Q}$  is another one with  $\|\dot{Q}\| < \delta$ , we also have

$$\delta - 5\delta \leq S(g, \dot{Q}) \leq \delta + 5\delta.$$

$$\text{Hence } |S(g, \dot{P}) - S(g, \dot{Q})| \leq (\delta + 5\delta) - (\delta - 5\delta) = 10\delta$$

$$\therefore \forall \varepsilon > 0, \exists \eta_\varepsilon = \frac{\varepsilon}{20} > 0 \text{ s.t.}$$

$$\text{if } \|\dot{P}\| < \eta_\varepsilon \text{ \& } \|\dot{Q}\| < \eta_\varepsilon,$$

$$\text{then } |S(g, \dot{P}) - S(g, \dot{Q})| \leq 10 \cdot \frac{\varepsilon}{20} = \frac{\varepsilon}{2} < \varepsilon$$

$\therefore$  Cauchy Criterion is satisfied.

(b) Applying Cauchy Criterion to show a function is not integrable:

$f$  is not integrable  $\Leftrightarrow \exists \varepsilon_0 > 0$ , s.t.  $\forall \eta > 0$ ,

$\exists \dot{\mathcal{P}}, \dot{\mathcal{Q}}$  with  $\|\dot{\mathcal{P}}\| < \eta$  &  $\|\dot{\mathcal{Q}}\| < \eta$  s.t.

$$|S(f, \dot{\mathcal{P}}) - S(f, \dot{\mathcal{Q}})| \geq \varepsilon_0$$

Concrete example:

Dirichlet function  $f(x) = \begin{cases} 1, & \text{if } x \text{ rational, } x \in [0, 1] \\ 0, & \text{if } x \text{ irrational, } x \in [0, 1]. \end{cases}$

(eg 5.1.6 (g), ex 12 of § 7.1)

Consider  $\varepsilon_0 = \frac{1}{2} > 0$ ,

$\forall \eta > 0$ , let  $\dot{\mathcal{P}} =$  any partition s.t.  $\|\dot{\mathcal{P}}\| < \eta$

with rational tags. (ie. all  $t_i \in \mathbb{Q} \cap [0, 1]$ )

$\dot{\mathcal{Q}} =$  any partition s.t.  $\|\dot{\mathcal{Q}}\| < \eta$

with irrational tags. (ie. all  $t_i \in [0, 1] \setminus \mathbb{Q}$ )

Then 
$$S(f, \dot{\mathcal{P}}) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) = \sum_{i=1}^n (x_i - x_{i-1}) = 1$$

$$S(f, \dot{\mathcal{Q}}) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) = 0$$

$$\Rightarrow |S(f, \dot{\mathcal{P}}) - S(f, \dot{\mathcal{Q}})| = 1 \geq \varepsilon_0$$

$\therefore f$  is not Riemann integrable.



Thm 7.2.3 (Squeeze Thm) let  $f: [a, b] \rightarrow \mathbb{R}$  ( $a < b$ )

Then  $f \in \mathcal{R}[a, b] \Leftrightarrow \forall \varepsilon > 0, \exists$  functions  $\alpha_\varepsilon$  and  $\omega_\varepsilon \in \mathcal{R}[a, b]$

with  $\alpha_\varepsilon(x) \leq f(x) \leq \omega_\varepsilon(x), \forall x \in [a, b]$

such that  $\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon$ .

(Remark = we don't need to assume  $\lim_{\varepsilon \rightarrow 0} \int_a^b \omega_\varepsilon$  or  $\lim_{\varepsilon \rightarrow 0} \int_a^b \alpha_\varepsilon$  exist,

but of course their existence follows from Thm 7.15(c))

Pf: ( $\Rightarrow$ ) If  $f \in \mathcal{R}[a, b]$ , take  $\alpha_\varepsilon \equiv f \equiv \omega_\varepsilon, \forall \varepsilon > 0$

Then  $\int_a^b (f - f) = 0 < \varepsilon$

( $\Leftarrow$ ) By assumption,  $\alpha_\varepsilon$  &  $\omega_\varepsilon \in \mathcal{R}[a, b]$ .

Hence  $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$  such that

if  $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$ , then  $|\mathcal{S}(\alpha_\varepsilon, \dot{\mathcal{P}}) - \int_a^b \alpha_\varepsilon| < \varepsilon$

and  $|\mathcal{S}(\omega_\varepsilon, \dot{\mathcal{P}}) - \int_a^b \omega_\varepsilon| < \varepsilon$

(This  $\delta_\varepsilon = \min\{\delta'_\varepsilon, \delta''_\varepsilon\} > 0$ , where  $\delta'_\varepsilon$  is for  $\alpha_\varepsilon$ ,  $\delta''_\varepsilon$  for  $\omega_\varepsilon$ )

Therefore  $\int_a^b \alpha_\varepsilon - \varepsilon < \mathcal{S}(\alpha_\varepsilon, \dot{\mathcal{P}})$

and  $\mathcal{S}(\omega_\varepsilon, \dot{\mathcal{P}}) < \int_a^b \omega_\varepsilon + \varepsilon$

Since  $\alpha_\varepsilon(x) \leq f(x) \leq \omega_\varepsilon(x)$ ,  $\forall x \in [a, b]$ , we have

$$S(\alpha_\varepsilon, \dot{\mathcal{P}}) \leq S(f, \dot{\mathcal{P}}) \leq S(\omega_\varepsilon, \dot{\mathcal{P}}).$$

$$\therefore \int_a^b \alpha_\varepsilon - \varepsilon < S(f, \dot{\mathcal{P}}) < \int_a^b \omega_\varepsilon + \varepsilon$$

Therefore, if  $\dot{\mathcal{P}}$  &  $\dot{\mathcal{Q}}$  are two tagged partitions with

$$\|\dot{\mathcal{P}}\| < \delta_\varepsilon \text{ and } \|\dot{\mathcal{Q}}\| < \delta_\varepsilon,$$

we have 
$$\int_a^b \alpha_\varepsilon - \varepsilon < S(f, \dot{\mathcal{P}}) < \int_a^b \omega_\varepsilon + \varepsilon$$

and 
$$\int_a^b \alpha_\varepsilon - \varepsilon < S(f, \dot{\mathcal{Q}}) < \int_a^b \omega_\varepsilon + \varepsilon$$

$$\begin{aligned} \Rightarrow |S(f, \dot{\mathcal{P}}) - S(f, \dot{\mathcal{Q}})| &< \left( \int_a^b \omega_\varepsilon + \varepsilon \right) - \left( \int_a^b \alpha_\varepsilon - \varepsilon \right) \\ &= \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) + 2\varepsilon \\ &< \varepsilon + 2\varepsilon = 3\varepsilon. \end{aligned}$$

Since  $\varepsilon > 0$  is arbitrary,  $f$  satisfies Cauchy Criterion

$$\therefore f \in \mathcal{R}[a, b] \quad \#$$

Recall (Def 5.4.9 of the Textbook)

A function  $\varphi: [a, b] \rightarrow \mathbb{R}$  is a step function

if  $\exists$  subintervals  $I_i$  (not necessary closed) with

$$\left\{ \begin{array}{l} I_i \cap I_j = \emptyset \text{ for } i \neq j \text{ and} \\ [a, b] = \bigcup_{i=1}^n I_i \end{array} \right.,$$

such that  $\varphi|_{I_i} = \text{const}$  function on  $I_i$ ,

i.e.  $\varphi(x) = k_i, \forall x \in I_i$  (for some  $k_i$ )

Lemma 7.2.4 Let  $\bullet J =$  subinterval of  $[a, b]$ ,

$\bullet c < d$  are endpoints of  $J$

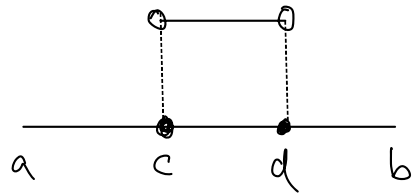
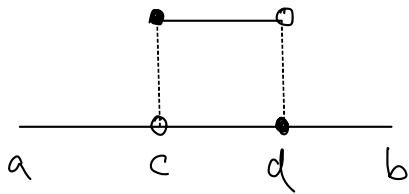
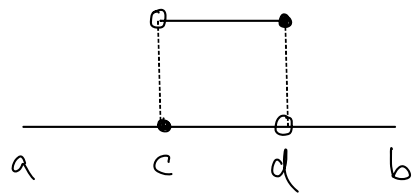
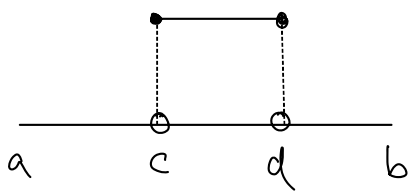
If 
$$\varphi_J(x) = \begin{cases} 1 & \text{for } x \in J \\ 0 & \text{for } x \notin J \text{ (} x \in [a, b] \text{)} \end{cases}$$

then  $\varphi_J \in \mathcal{R}[a, b]$  and  $\int_a^b \varphi_J = d - c$ .

Pf: There are 4 cases for  $J$ :

$J = [c, d], (c, d], [c, d)$  and  $(c, d)$

and corresponding 4 cases of  $\varphi_J$



All these 4 cases different from each others by finitely many points (at most 2), therefore all 4 cases have the same integral by Thm 7.1.3

By Ex 7.1.13 (Homework 4), we have

$$\int_a^b \varphi_J = d - c \quad \text{for the case of } J = [c, d].$$

Hence  $\int_a^b \varphi_J = d - c$  for all cases. ~~✗~~

Thm 7.2.5 If  $\varphi: [a, b] \rightarrow \mathbb{R}$  is a step function, then  $\varphi \in \mathcal{R}[a, b]$

(i.e. step functions are Riemann integrable)

Pf: Assume  $\varphi(x) = k_i$  for  $x \in I_i$

$$\left( I_i \cap I_j = \emptyset, \bigcup_{i=1}^n I_i = [a, b] \right)$$

Then by using the notations in Lemma 7.2.4,

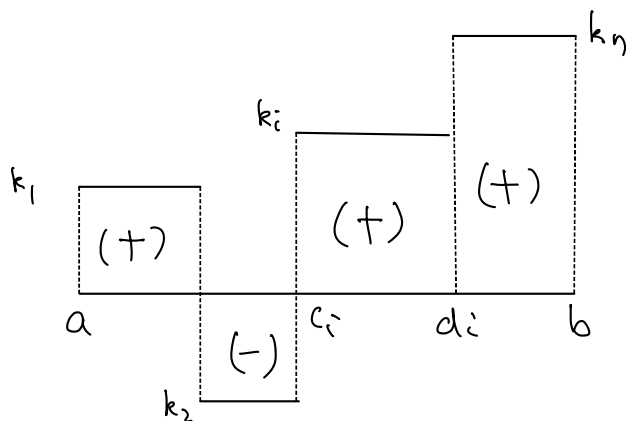
$$\varphi(x) = \sum_{\bar{x}=1}^n k_{\bar{x}} \varphi_{I_{\bar{x}}}(x).$$

Since  $\varphi_{I_{\bar{x}}} \in \mathcal{R}[a,b]$ , Thm 7.1.5 (a) & (b)  $\Rightarrow \varphi \in \mathcal{R}[a,b]$  ~~\*\*\*~~

Remark: Moreover,

if  $I_{\bar{x}} = [c_{\bar{x}}, d_{\bar{x}}]$ ,  $\bar{x}=1, \dots, n$ , then Lemma 7.2.4  $\Rightarrow$

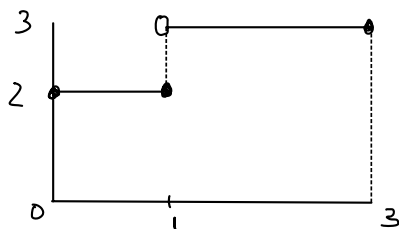
$$\int_a^b \varphi = \sum_{\bar{x}=1}^n k_{\bar{x}} \int_a^b \varphi_{I_{\bar{x}}} = \sum_{\bar{x}=1}^n k_{\bar{x}} (d_{\bar{x}} - c_{\bar{x}}).$$



Eg 7.2.6

(a) (Eg 7.1.4 (b) again)

$g: [0,3] \rightarrow \mathbb{R}$  defined by  $g(x) = \begin{cases} 3, & 1 < x \leq 3 \\ 2, & 0 \leq x \leq 1 \end{cases}$



is a step function.

Clearly  $g(x) = 2 \varphi_{[0,1]}(x) + 3 \varphi_{(1,3]}(x) \in \mathcal{R}[0,3]$

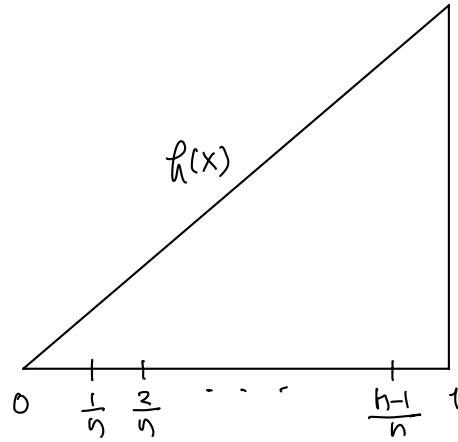
$$\begin{aligned} \text{and } \int_a^b g &= 2 \int_a^b \varphi_{[0,1]} + 3 \int_a^b \varphi_{(1,3]} \\ &= 2 \cdot (1-0) + 3 \cdot (3-1) = 8 \quad \# \end{aligned}$$

(b) (eg 7.1.4 (c))

$f(x) = x$  on  $[0,1]$ .

Consider (uniform) partition

$$\mathcal{P}_n = \left\{ \left[ \frac{k-1}{n}, \frac{k}{n} \right] \right\}_{k=1}^n$$

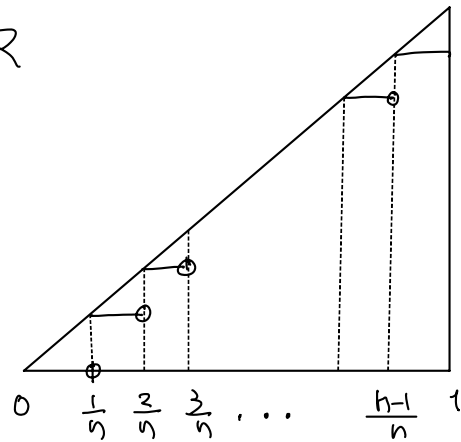


Define step functions  $\alpha_n: [0,1] \rightarrow \mathbb{R}$

by (for  $k=1, \dots, n$ )

$$\alpha_n(x) = \frac{k-1}{n} \quad \text{for } x \in \left[ \frac{k-1}{n}, \frac{k}{n} \right),$$

$$\text{and } \alpha_n(1) = \frac{n-1}{n}$$

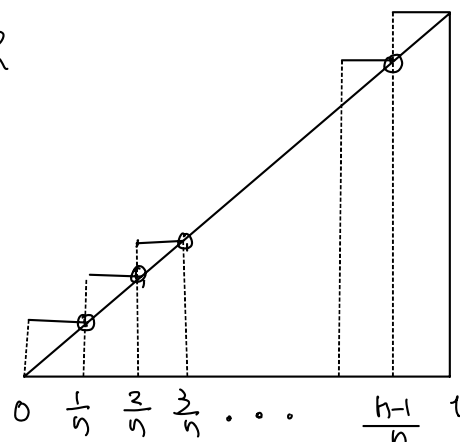


Define step functions  $\omega_n: [0,1] \rightarrow \mathbb{R}$

by (for  $k=1, \dots, n$ )

$$\omega_n(x) = \frac{k}{n} \quad \text{for } x \in \left[ \frac{k-1}{n}, \frac{k}{n} \right),$$

$$\text{and } \omega_n(1) = 1$$



Then clearly  $\alpha_n(x) \leq f(x) \leq \omega_n(x) \quad \forall x \in [0, 1]$ ,

$$\text{as } \alpha_n(x) = \min_{[\frac{k-1}{n}, \frac{k}{n}]} f(x) \quad \& \quad \omega_n(x) = \max_{[\frac{k-1}{n}, \frac{k}{n}]} f(x)$$

$$\text{for } x \in [\frac{k-1}{n}, \frac{k}{n}) \quad \left( [\frac{n-1}{n}, 1] \text{ for } k=n \right)$$

By Thm 7.2.5

$$\begin{aligned} \int_0^1 \alpha_n &= 0 \cdot \frac{1}{n} + \frac{1}{n} \cdot \frac{1}{n} + \frac{2}{n} \cdot \frac{1}{n} \cdots + \frac{(n-1)}{n} \cdot \frac{1}{n} \\ &= \frac{1}{n^2} (1 + \cdots + (n-1)) = \frac{1}{n^2} \cdot \frac{(n-1)n}{2} \\ &= \frac{1}{2} \left(1 - \frac{1}{n}\right) \end{aligned}$$

$$\begin{aligned} \int_0^1 \omega_n &= \frac{1}{n} \cdot \frac{1}{n} + \frac{2}{n} \cdot \frac{1}{n} + \frac{3}{n} \cdot \frac{1}{n} \cdots + \frac{n}{n} \cdot \frac{1}{n} \\ &= \frac{1}{n^2} (1 + \cdots + n) = \frac{1}{n^2} \frac{n(n+1)}{2} \\ &= \frac{1}{2} \left(1 + \frac{1}{n}\right) \end{aligned}$$

Hence (by Thm 7.1.5)

$$\int_0^1 (\omega_n - \alpha_n) = \frac{1}{n}$$

$\therefore \forall \varepsilon > 0$ , choose  $n_\varepsilon$  st.  $\frac{1}{n_\varepsilon} < \varepsilon$ , then

$$\alpha_{n_\varepsilon}(x) \leq f(x) \leq \omega_{n_\varepsilon}(x) \quad \text{st.}$$

$$\int_0^1 (\omega_{n_\varepsilon} - \alpha_{n_\varepsilon}) = \frac{1}{n_\varepsilon} < \varepsilon.$$

By Squeeze Thm (7.2.3),  $f(x) = x \in \mathcal{R}[0,1]$ .

Furthermore, by Thm 7.1.5

$$\frac{1}{2} \left(1 - \frac{1}{n_\varepsilon}\right) = \int_0^1 \alpha_{n_\varepsilon} \leq \int_0^1 f \leq \int_0^1 \omega_{n_\varepsilon} = \frac{1}{2} \left(1 + \frac{1}{n_\varepsilon}\right)$$

Letting  $\varepsilon \rightarrow 0$ , we have  $n_\varepsilon \rightarrow \infty$ , and hence

$$\int_0^1 f = \frac{1}{2} \quad \ast$$

Thm 7.2.7 If  $f: [a,b] \rightarrow \mathbb{R}$  is continuous, then  $f \in \mathcal{R}[a,b]$ . ( $-\infty < a < b < +\infty$ )

(Continuous functions on closed & bounded interval are Riemann integrable)

Pf: By Thm 5.4.3, cont. functions on closed & bounded interval are uniformly continuous.

$\therefore \forall \varepsilon > 0$ ,  $\exists \delta_\varepsilon > 0$  (indep. of points) such that

if  $|x - y| < \delta_\varepsilon$  ( $x, y \in [a,b]$ ),

then  $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$

Then take any partition  $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^n$  such that

$\|\mathcal{P}\| < \delta_\varepsilon$ . (such  $\mathcal{P}$  always exists)



Since  $f$  is continuous

$$\exists x'_i \in [x_{i-1}, x_i] \text{ such that } f(x'_i) = \min_{[x_{i-1}, x_i]} f(x) \text{ and}$$

$$\exists x''_i \in [x_{i-1}, x_i] \text{ such that } f(x''_i) = \max_{[x_{i-1}, x_i]} f(x)$$

Define step functions

$$\alpha_\varepsilon(x) = \begin{cases} f(x'_i) & \text{for } x \in [x_{i-1}, x_i) \text{ for } i \neq n \\ f(x'_n) & \text{for } x \in [x_{n-1}, x_n] \end{cases}$$

and

$$\omega_\varepsilon(x) = \begin{cases} f(x''_i) & \text{for } x \in [x_{i-1}, x_i) \text{ for } i \neq n \\ f(x''_n) & \text{for } x \in [x_{n-1}, x_n] \end{cases}$$

$$\text{Then } \alpha_\varepsilon(x) \leq f(x) \leq \omega_\varepsilon(x) \quad \forall x \in [a, b].$$

Moreover,

$$\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) = \sum_{i=1}^n (f(x''_i) - f(x'_i)) (x_i - x_{i-1})$$
$$< \sum_{i=1}^n \left( \frac{\varepsilon}{b-a} \right) (x_i - x_{i-1})$$

$$\text{Since } x''_i, x'_i \in [x_{i-1}, x_i]$$

$$\Rightarrow |x''_i - x'_i| \leq x_i - x_{i-1} \leq \|\mathcal{P}\| < \delta_\varepsilon.$$

$$\therefore \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \frac{\varepsilon}{b-a} \cdot (b-a) = \varepsilon$$

Hence Squeeze Thm (7.2.3)  $\Rightarrow f \in \mathcal{R}[a, b]$  ~~✗~~